Mechanical Vibrations

VIBRATION

Single DoF free vibration system

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Single DoF free vibration system

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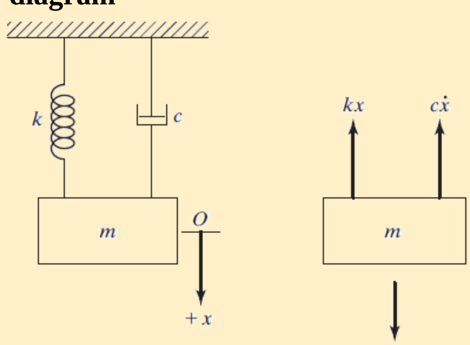
Single DoF free damped vibration system

System free body diagram

C is the damping constant or coefficient



System　　　Free-body diagram

Single DoF free vibration system

Single DoF free damped vibration system

Mathematical model (governing equation)

$$m\ddot{x}+c\dot{x}+kx=0$$

Solution:

$$ms^2+cs+k=0 \quad \Rightarrow \quad s_{1,2}=\frac{-c\pm\sqrt{c^2-4mk}}{2m}$$

Characteristic equation

$$s_{1,2}=-\frac{c}{2m}\pm\sqrt{\left(\frac{c}{2m}\right)^2-\frac{k}{m}}$$

Single DoF free vibration system

Single DoF free damped vibration system

Critical damping (c_c)

The critical damping c_c is defined as the value of the damping constant c for which the radical in s-equation becomes zero:

$$\left(\frac{c_c}{2m}\right)^2-\frac{k}{m}=0\Rightarrow c_c=2m\sqrt{\frac{k}{m}}=2m\omega_n$$

damping ratio (ζ)

$$\xi=\frac{c}{c_c}=\frac{c}{2m\omega_n}$$

So:

$$s_{1,2}=\left(-\xi\pm\sqrt{\xi^2-1}\right)\omega_n$$

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Single DoF free damped vibration system

$$s_{1,2}=\left(-\xi\pm\sqrt{\xi^2-1}\right)\omega_n$$

Case 1: under-damped system (ζ<1)

$$s_{1,2}=\left(-\xi\pm i\sqrt{1-\xi^2}\right)\omega_n$$

$$x(t)=e^{-\xi\omega_n t}\left\{C_1\cos\left(\sqrt{1-\xi^2}\,\omega_n t\right)+C_2\sin\left(\sqrt{1-\xi^2}\,\omega_n t\right)\right\}$$

$$x(t)=X_o e^{-\xi\omega_n t}\left\{\sin\left(\sqrt{1-\xi^2}\,\omega_n t+\phi_o\right)\right\}$$

$$x(t)=Xe^{-\xi\omega_n t}\left\{\cos\left(\sqrt{1-\xi^2}\,\omega_n t-\phi\right)\right\}$$

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Case 1: under-damped system (ζ<1)

Assume I.Cs:

$$x(t=0)=x_o=C_1$$

$$\dot{x}(t=0)=\dot{x}_o\Rightarrow C_2=\frac{\dot{x}_o+\xi\omega_n x_o}{\sqrt{1-\xi^2}\,\omega_n}$$

$$x(t)=e^{-\xi\omega_n t}\left\{x_o\cos\left(\sqrt{1-\xi^2}\,\omega_n t\right)+\frac{\dot{x}_o+\xi\omega_n x_o}{\sqrt{1-\xi^2}\,\omega_n}\sin\left(\sqrt{1-\xi^2}\,\omega_n t\right)\right\}$$

Single DoF free vibration system

Case 1: under-damped system (ζ<1)

The constants *X*, *X*

The constants X, X_o, ϕ and ϕ_o can be expressed as:

$$X = X_o = \sqrt{C_1^2 + C_2^2} = \frac{\sqrt{x_o^2\omega_n^2 + \dot{x}_o^2 + 2x_o\dot{x}_o\xi\omega_n}}{\sqrt{1-\xi^2}\,\omega_n}$$

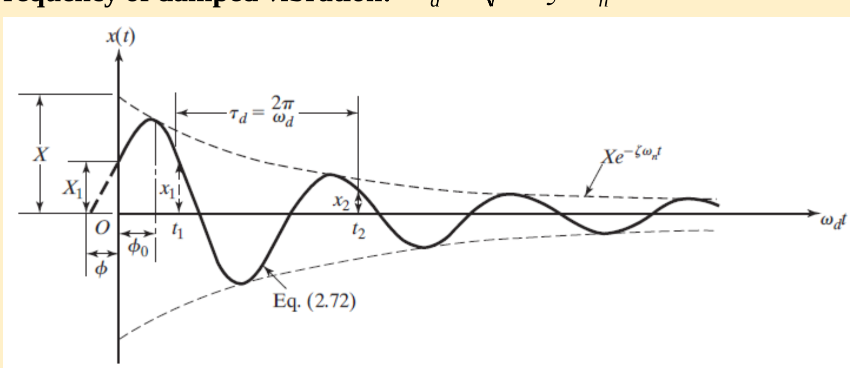
$$\phi_o = \tan^{-1}\left[\frac{C_1}{C_2}\right] = \tan^{-1}\left[\frac{x_o\omega_n\sqrt{1-\xi^2}}{\dot{x}_o + \xi\omega_n x_o}\right]$$

$$\phi = \tan^{-1}\left[\frac{C_2}{C_1}\right] = \tan^{-1}\left[\frac{\dot{x}_o + \xi\omega_n x_o}{x_o\omega_n\sqrt{1-\xi^2}}\right]$$

Single DoF free vibration system

Case 1: under-damped system (ζ<1)

Frequency of damped vibration: $\omega_d = \sqrt{1-\xi^2}\,\omega_n$



Under-damped solution

Single DoF free vibration system

Single DoF free damped vibration system

Case 2: critically damped system (ζ=1)

$$s_{1,2} = -\omega_n$$

$$x(t) = e^{-\omega_n t}\{C_1 + C_2 t\}$$

$$x(t=0) = C_1 = x_o$$

$$\dot{x}(t=0) = \dot{x}_o \Rightarrow C_2 = \dot{x}_o + \omega_n x_o$$

$$x(t) = e^{-\omega_n t}\left\{x_o + \left(\dot{x}_o + \omega_n x_o\right)t\right\}$$

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Case 3: overdamped system (ζ>1)

$$s_{1,2} = \left(-\xi \pm \sqrt{\xi^2 - 1}\right)\omega_n < 0$$

$$x(t) = C_1 e^{-\left(\xi + \sqrt{\xi^2 - 1}\right)\omega_n t} + C_2 e^{-\left(\xi - \sqrt{\xi^2 - 1}\right)\omega_n t}$$

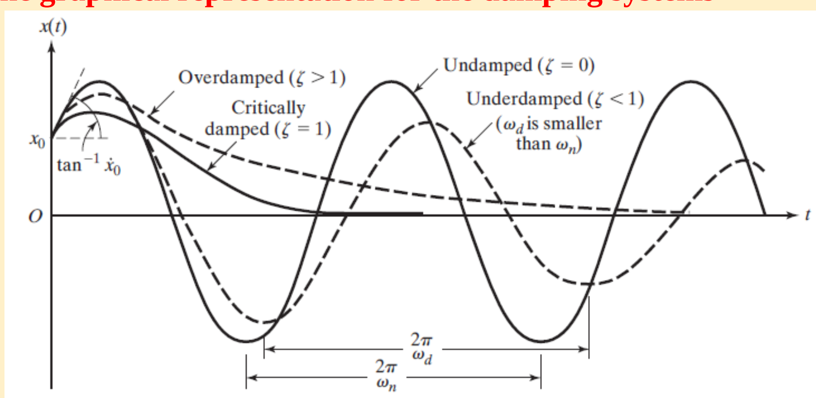
$$x(t=0) = x_o \Rightarrow C_1 = \frac{x_o \omega_n \left(\xi + \sqrt{\xi^2 - 1}\right) + \dot{x}_o}{2\omega_n \sqrt{\xi^2 - 1}}$$

$$\dot{x}(t=0) = \dot{x}_o \Rightarrow C_2 = \frac{-x_o \omega_n \left(\xi - \sqrt{\xi^2 - 1}\right) - \dot{x}_o}{2\omega_n \sqrt{\xi^2 - 1}}$$

Single DoF free vibration system

Single DoF free damped vibration system

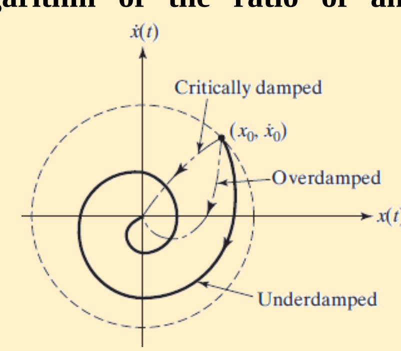
The graphical representation for the damping systems



Single DoF free vibration system

Logarithmic decrement

The logarithmic decrement represents the rate at which the amplitude of a free-damped vibration decreases. It is defined as the natural logarithm of the ratio of any two successive amplitudes.



Phase plane of a damped system.

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Logarithmic decrement

$$\frac{x_1}{x_2}=\frac{X_o e^{-\xi\omega_n t_1}\cos(\omega_d t_1-\phi_o)}{X_o e^{-\xi\omega_n t_2}\cos(\omega_d t_2-\phi_o)}$$

But

$$t_2=t_1+\tau_d \Rightarrow \tau_d=\frac{2\pi}{\omega_d}$$

$$\cos(\omega_d t_2-\phi_o)=\cos(2\pi+\omega_d t_1-\phi_o)=\cos(\omega_d t_1-\phi_o)$$

So

$$\frac{x_1}{x_2}=\frac{e^{-\xi\omega_n t_1}}{e^{-\xi\omega_n(t_1+\tau_d)}}=e^{\xi\omega_n\tau_d}$$

Assume: δ the logarithmic decrement

$$\delta=\ln\frac{x_1}{x_2}=\xi\omega_n\tau_d=\xi\omega_n\frac{2\pi}{\sqrt{1-\xi^2}\,\omega_n}=\frac{2\pi\xi}{\sqrt{1-\xi^2}}=\frac{2\pi}{\omega_d}\cdot\frac{c}{2m}$$

Logarithmic decrement: is dimensionless

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VIBRATION

Logarithmic decrement

$$\xi=\frac{\delta}{\sqrt{(2\pi)^2+\delta^2}}$$

For small damping ; ζ <<1

$$\delta\approx 2\pi\xi$$

Single DoF free vibration system

V I B R A T I O N S

Example 2.6

An under-damped shock absorber is to be designed for a motorcycle of mass 200 kg (Fig.(a)).When the shock absorber is subjected to an initial vertical velocity due to a road bump, the resulting displacement-time curve is to be as indicated in Fig.(b).

Requirements:

1. Find the necessary stiffness and damping constants of the shock absorber if the damped period of vibration is to be 2s and the amplitude x_1 is to be reduced to one-fourth in one half cycle (i.e. $x_{1.5} = x_1/4$).

Single DoF free vibration system

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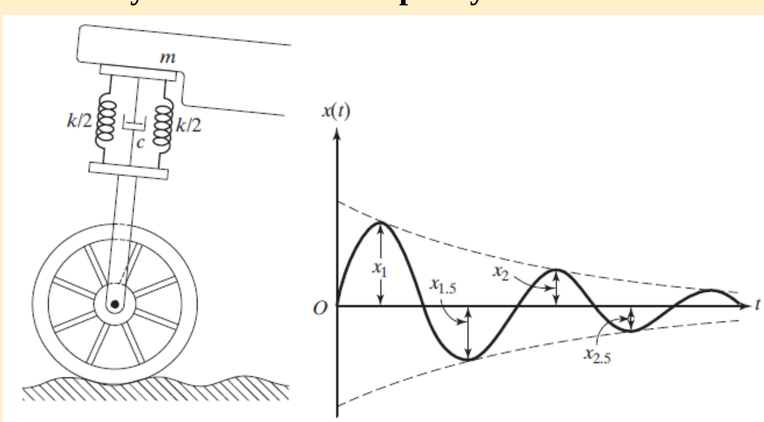
Example 2.6

Note that this system is under-damped system



(a) (b)

Single DoF free vibration system

VIBRATIONS

Example 2.6

Solution:

Finding k and c

Since $x_{1.5}$ = $x_1/4$, $x_2 = x_{1.5}/4$ = $x_1/16$ and so, the logarithmic decrement become:

$$\delta = \ln\frac{x_1}{x_2} = \ln(16) = 2.7726 = \frac{2\pi\xi}{\sqrt{1-\xi^2}}$$

$$\Rightarrow \xi = 0.4037$$

Hence:

$$\tau_d = 2 = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n\sqrt{1-\xi^2}} \Rightarrow \omega_n = \frac{2\pi}{\omega_n\sqrt{1-0.4037^2}} = 3.4338\,rad/s$$

Single DoF free vibration system

VIBRATIONS

Example 2.6

Solution:

Finding k and c [cont]

The critical damping can be found as:

$$c_c = 2m\omega_n = 2(200)(3.4338) = 1373.54\ \text{N-s/m}$$

And so the damping constant can be found as:

$$c = \xi\, c_c = (0.4037)(1373.54) = 554.4981\,\text{N-s/m}$$

And stiffness can be found as:

$$k = m\omega_n^2 = (200)(3.4338)^2 = 2358.2652\,\text{N/m}$$

Single DoF free vibration system

STATICS

End of chapter